# Reduction of Computational Time for a Robust Kalman Filter Through Leroux Gueguen Algorithm

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***Abstract-***Data loss is a frequent dilemma in many processes including state estimation. These lost data samples are normally reconstructed by employing linear prediction theory. Three various linear prediction schemes that includes Normal Equation (NE), Levinson-Durbin Algorithm (LDA) and Leroux Gueguen Algorithm (LGA) may be employed to reconstruct the data loss. The NE method suffers from high computational complexity. On the other hand, LDA is computationally less expensive but it has large dynamic range problem in Linear Prediction Coefficients (LPCs). The LGA overcomes the drawbacks associated with NE and LDA schemes. The major contribution of this paper is the reduction of computational time raised by NE method by employing a modified LGA technique. The upper limit of linear prediction filter order is decided by a minimum mean square error based algorithm. The simulation results are shown by employing this modified LGA on a standard Mass-Spring-Damper system.

***Keywords-***Linear Prediction Schemes, Kalman filter, Leroux Gueguen Algorithm, Linear Prediction Coefficients, Normal Equation

## I. INTRODUCTION

Linear prediction is in fact a system identification process where a signal is reconstructed from its previous signal samples [i]. In other words, linear prediction is a mathematical and intensification tool for estimating the future values of a signal based on its previous values (and sometimes input as well). The theory of linear prediction has been extensively used in a variety of engineering applications [ii]. Its diverse range of applications can be found in speech coding, speed recognition, model based spectral analysis, signal restoration, video coding, model based interpolation and impulse/step input detection [iii-iv].

In linear prediction a signal window comprised of previous samples is selected to reconstruct the lost data. In order to minimize the mean square error, the weights are assigned according to their contribution to this data [v]. These weights are called linear prediction coefficients. Three linear prediction coefficient techniques namely Normal Equation, Levinson Durbin Algorithm and Leroux Gueguen Algorithm may be employed to reconstruct the lost data. During the last decade, overcoming the side issues emerged from missing data in control and communication systems are prevailed as open research problems for researchers [vi].

Perhaps, the best available tool for the linear estimation problem is Kalman filtering. Kalman filter performs estimation based on noisy measurement data and input. Normally in Kalman filter there are two steps; (a) predicts the states of a system, (b) then update the states using measured data. In case of loss of measured data the update step will not be performed and the estimation of conventional Kalman filter may not be accurate. To remove this drawback, an alternative method Open-loop Estimation is used to estimate the state of a system [vii]. In Open-loop Kalman filter scheme only prediction is performed in the absence of data. And when the data reoccurs then update is performed. However this scheme produces unbounded estimation error, when the data loss occurs for an adequate time period [viii-ix]. To estimate the state of a system more accurately, an optimal estimation techniques is required, which can reduce the estimation error to its bound in case of data loss [vi].

As mentioned above, there are three LPCs techniques that are used to reconstruct the missing measurements. The NE method has been found computationally expensive due to involving larger matrix inversion in calculating LPCs [v]. The LDA avoids the matrix inversions involved in conventional Normal Equation method and hence it reduces the computational cost. However, LDA suffers from large dynamic variety in the values of LPCs [x]. Theoretically speaking, it is observed that no limit can be made on the value of LPCs computed through LDA [xi].

On the other hand, LGA removes the issues associated with LDA in a fixed-field by using the application of Schwartz inequality in computing LPCs through this scheme [x]. The LGA method also avoids the inversion of large matrices involved in NE, hence reduces computational time. This paper emphasis on the implementation of LGA for a Mass-Spring-Damper system in order to (a) test the performance of LGA and(b) provide another platform of handling data. The LerouxGueguen Algorithm is a better solution in a non-variable environment since the scale of intermediate variables is bounded [xii]. The core objective of this paper is to reduce the computational time in calculating LPCs through LGA as compared to NE method.

The rest of paper is organized as follows: In Section II, an overview of discrete-time Kalman filtering is given. Section III presents the existing solution to compensate loss of observation in KF. Section IV discusses linear prediction theory wherein linear prediction schemes (NE and LGA) are discussed. The third linear prediction technique, LDA is not discussed in this paper in order to focus attention on the core contribution of implementing LerouxGueguen Algorithm. In Section V, a numerical example of MSD system,its dynamics and also simulation and results are shown. The paper is concluded with suggestions in the last section.

## II. Discrete Time Kalman Filtering

Consider following discrete LTI system

$$x_{k+1} = Ax_k + Bu_k + w_k \tag{1}$$
$$z_k = Cx_k + v_k \tag{2}$$

In the above equations, $k \in R = \{0,1,2,.....\}$ is the discrete time instant, $x \in R^l$ is the input signal, z is the measurement noise, v is the sensor noise, $A \in R^{nxn}$ is the state transition matrix, $B \in R^{nxl}$ is the input matrix, $C \in R^{mxn}$ is the output matrix and $(x_0, w_k, v_k)$ and uncorrelated Gaussian white noise sequences with mean $(x_0, 0, 0)$ and covariance $(P_0, Q_k, R_k)$.

Algorithm 1: Basic Kalman filter

1. Initialize
 $x_{0|0}, u_0, w_0, v_0 P_{0|0}$ *and* $k = 1$
2. Prediction step
 $x_{k+1|k} = Ax_{k|k} + Bu_k$ ; State estimation
 $P_{k+1|k} = AP_{k|k}A^T + Q_k$ ;Covariance Estimation
3. Update of Time-step
 $$k \to (k+1)$$
4. Observation Obtained
 $$z_{k+1|k} = Cx_{k+1} - v_{k+1}$$
 Compute the innovation vector
 $$r_{k+1|k} = z_{k+1} - Cx_{k+1|k}$$
 Compute the innovation Cov. Matrix
 $$S_{k+1|k} = CP_{k+1|k}C^T + R_{k+1}$$
 Compute the Kalman filter gain equation
 $$k_{k+1} = P_{k+1|k}C^T S^{-1}{}_{k+1}$$
5. Update cycle
 $x_{k+1|k+1} = x_{k+1|k} + k_{k+1}r_{k+1}$ ; state Estimation
 $P_{k+1|k+1} = (1 - k_{k+1}C)P_{k+1|k}$ ; Cov.Estimation
6. Return to step 2

The estimation through Kalman filter is summarized in the Algorithm 1. From the above Algorithm it is easy to realize the update step is totally dependent on measurements. When output data ($z_k$) is unavailable, Kalman filter may not result in optimal estimation. In such a situation, we used three different linear prediction methods for compensating the missing output for update step.

## III. Open Loop Kalman Filtering

In simple words, in OLE when measurement data is not available the Kalman filter gain is set to zero, which means that no update cycle is carried out as long as data is unavailable. Only prediction step is performed repeatedly, and when the data reoccurs update step will be executed. In OLKF, prediction is referred as "estimation" [xiii]. OLE has simpler structure, so it takes much less time in estimation, but it has some disadvantages, which are given below.

1) OLE technique may diverge in practice when data loss occurs for long time and it is likely that error covariance could exceed the limit/bounds if the upper and lower bounds of error covariance are provided [viii].
2) When measured data becomes available after the loss period, oscillations and/or sharp spikes can be observed in the estimated parameters [xiii].
3) The steady state values of state and covariance are not regained even after recovery of data loss. It takes too longer to approach the steady state [xiv].

Open-Loop Estimation algorithm is summarized as follows.

Algorithm 2: Open Loop KF

1. Initialize
 $x_{0|0}, u_0, w_k, v_k P_{0|0}$ *and* $k = 1$
2. Prediction cycle
 $x_{k+1|k} = Ax_{k|k} + Bu_k$ ; state estimation
 $P_{k+1|k} = AP_{k|k}A^T + Q_k$ ; Cov. Estimation
3. Time step update
 $$k \to (k+1);$$
4. Observation Obtained
 $z_{k+1}$ is not available
 There is no residual innovation and hence Kalman gain is not calculated.
 $x_{k+1|k+1} \leftarrow x_{k+1|k}$; State Estimation
 $P_{k+1|k+1} \leftarrow P_{k+1|k}$; Cov.Estimation

5. Return to step 2

Due to the aforementioned limitations, a robust technique is required to improve estimation with limited error covariance in case of data loss. In order to suit the problem of data loss, the existing linear prediction methods [i] are amended by providing minimum mean square error based algorithm that decides the upper bound of filter order.

## IV. LINEAR PREDICTION METHODS

According to linear prediction theory the future values of a discrete time signal are estimated as a Linear combination of the present and past samples of the signal. The missing samples can be reconstructed from its previous $M$ samples using the following equation.

$$\bar{z}_k = \sum_{i=1}^{M} \alpha_i z_{k-i} \tag{3}$$

where $\bar{z}_k$ is the measured signal or compensated observation, $M$ represents Linear Prediction Filter Order and parameter represents the weights assigned to the previous observations according to their contribution and are known as LPCs. LPCs can be computed using various methods but some optimal methods are Normal Equation, Levinson Durbin Algorithm and LeurouxGeugeun Algorithm. In routine practice, no strategy is available to decide the value of `M` in NE, LDA and LGA. However, some theoretical bound is always required to decide the threshold limit on the value of `M`. Hence, the word "Modified" is frequently adopted to these schemes such as one is shown in Algorithm 3.

Algorithm 3: Choice of the LPFO

1. *Trace* $E_{max}(k-1) = \max(E_i)$, *where* $E_i = |x_i - \hat{x}_i|_2 \quad i = \{1,2,\dots\ k-1\}(state\ vector)$.
2. *Initialization* $n = 1$, *Compute* $R_\psi$ *and* $r_\psi$.
3. Recursion = 2, ...,M
   *Obtain* $\bar{z}$ using Equation 3.
   *Calculate* measurement updated state estimation $\hat{x}_k$ based on this compensated observations.
   *Compute* $E_n(k) = |x_k - \hat{x}_k|_2$.
   *Check* Is $E_n(k) \leq E_{max}(k-1)$
   *Yes* $M \leftarrow n$ :order of the LP filter
   *Otherwise* $n \leftarrow n+1$
4. Repeatstep 3.

*a. Normal Equation Method*

The Normal Equation derivation is actually based on the minimization of the mean square error. In Normal Equation method LPCs $\alpha_i$ are calculated as

$$A_\alpha = r_\varphi R_\varphi^{-1} \tag{4}$$

where

$$R_\varphi = \begin{pmatrix} R_\varphi(0) & R_\varphi(1) & R_\varphi(2) & \cdots & R_\varphi(M-1) \\ R_\varphi(1) & R_\varphi(0) & R_\varphi(1) & \cdots & R_\varphi(M-2) \\ R_\varphi(2) & R_\varphi(1) & R_\varphi(0) & \cdots & R_\varphi(M-3) \\ & \vdots & & \ddots & \vdots \\ R_\varphi(M-1) & R_\varphi(M-2) & R_\varphi(M-3) & \cdots & R_\varphi(0) \end{pmatrix}$$

is the authocorrelated matrix,

$$A_\alpha = [\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \cdots \quad \alpha_n]^T \tag{5}$$

the desired LPCs array,

$$r_\varphi = [r_\varphi(1) \quad r_\varphi(2) \quad r_\varphi(3) \quad \cdots \quad r_\varphi(M)]^T \tag{6}$$

The autocorrelation array with

$$E(z_{k-i}z_{k-n}) = \begin{cases} R_\varphi(0) & if\ i = n \\ R_\varphi(|i-n|) & if\ i \neq n \end{cases} \tag{7}$$

and

$$r_\varphi(n) = E(z_k z_{k-n}) \tag{8}$$

From equation (4), the optimal values of the modified LPCs are computed.

*b. Modified LerouxGueguen Algorithm*

As discussed before, the earlier schemes, namely OLE, modified NE and modified LDA have their own limitations. Therefore, a strategy is required which could answer to these limitations handsomely. In this paper LerouxGueguen Algorithm has been believed to overcome these limitations. In order to reconstruct the data required in measurement update

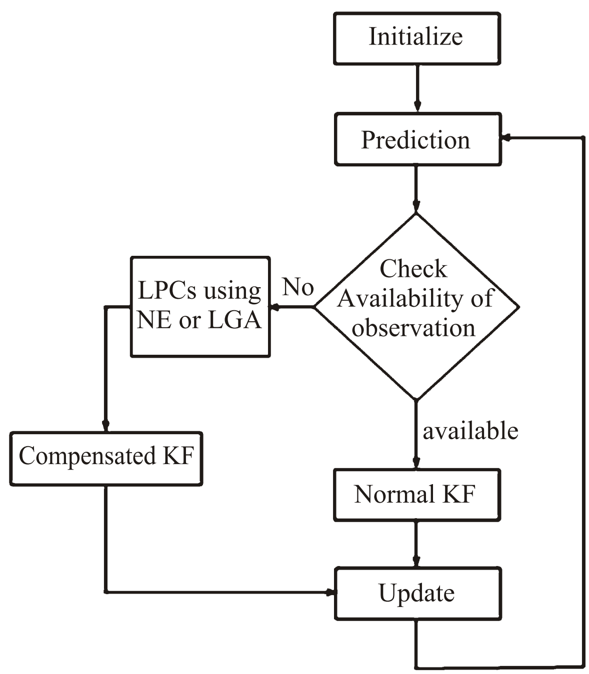


Fig. 1. Flow chart of the proposed scheme

Algorithm 4: Modified Leroux Gueguon Algorithm

1. *Initialization* $x_{0|0}, i = 0;\ \zeta^{0}[k] = R[k] for\ k = -M + i, \dots, M$

2. *Threshold Error*
   Set the value of threshold error $E_{th}$.

3. Recursion i={1,2,3,…, M} where M=LPFO
   3.1 Compute the $i^{th}$ reflection coefficients as
   $$\bar{k}_i = \frac{\zeta^{(i-1)}[i]}{\zeta^{(i-1)}[0]}$$
   Stop when $i = M$.
   3.2 a) Compute the error signal $E_i = |x_i - \hat{x}_i|_2$
   b) Is $E_i \leq E_{th}$
   Yes; stop the process **M←I**
   3.3 Compute the values of $\zeta$ parameters
   $$\zeta^{(i)}[i] = \varepsilon^{(i-1)}[k] - \bar{k}_i \zeta^{(i-1)}[i-k];$$
   $$K = -M + i + 1, \dots, 0, \dots, M$$
   Update $i \leftarrow i + 1$ and
   Repeat Step 3.1

step of Kalman filter, LGA is integrated with the Kalman filtering process. It computes reflection coefficients without dealing directly with LPCs from the autocorrelation matrix. LGA also reduces the computational cost of NE method. A Modified LGA is given in following algorithm.

The values of are used to calculate the LPCs. As the intermediate variables having bounded values so LGA technique depicts better performance in fixed-point environment. LGA has also a problem that it returns only RCs, which is not a major concernif the filter is in lattice form [x]. In order to test the performances of NE and LGA (i.e how better these techniques could reconstruct the data), these schemes are integrated with the conventional Kalman filter as shown in the subsequent section.

In the following section, the theory presented for proposed schemes is tested on a well-known numerical example of Mass-Spring-Damper system.

## V. Numerical Example of Mass-Spring-Damper System

The section presents 1) the dynamics of a simple second order mechanical system i.e. Mass-Spring-Damper system and 2) the simulation results for the three under discussion schemes.

### *A. Dynamic of MSD*

In this subsection, a simple second order mechanical system namely Mass-Spring-Damper system is demonstrated. Such systems are commonly control experimental devices frequently encountered in many technical laboratories [xiv].

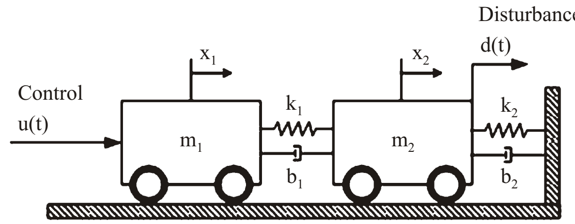


Fig. 2. Mass-Spring-Damper system

In the Fig. 2, *u(t)* is control input, $k_1$ and $k_2$ are spring constants, $b_1$ and $b_2$ are coefficients of viscous damping, $m_1$ and $m_2$ are masses, and $x_1$ and $x_2$ are the displacements of masses $m_1$ and $m_2$ respectively. The dynamics of MSD system are described by the following mathematical equations:

$$\dot{x}(t) = Ax(t) + Bu(t) + Hw(t) z(t) = Cx(t) + v(t) \quad (9)$$

where the state vector consists of displacement and speed of two masses $m_1$ and $m_2$ and is described as

$$x(t) = [x_1(t) \quad x_2(t) \quad v_1(t) \quad v_2(t)]^T$$

and

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_1}{m_1} & \frac{k_1}{m_1} & -\frac{b_1}{m_1} & \frac{b_1}{m_1} \\ \frac{k_1}{m_2} & -\frac{k_1 + k_2}{m_2} & \frac{b_1}{m_1} & -\frac{b_1 + b_2}{m_2} \end{bmatrix}$$

$$B = [0 \quad 0 \quad \frac{1}{m_1} \quad 0]^T$$

$$C = [0 \quad 1 \quad 0 \quad 0]$$

$$H = [0 \quad 0 \quad 0 \quad 3] \quad (10)$$

The MSD system disturbance and sensor noise dynamics are described as *E[w(t)]=0*, *E[v(t)]=0*. By substituting the values of parameters $m_1 = m_2 = 1$, $k_1 = 1$, $k_2 = 0.15$, $b_1 = b_2 = 0.1$ and sampling interval is $T_s = 1ms$. Hence, the above matrices will become

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & -0.1 & 0.1 \\ 0.1 & -1.15 & 0.1 & -0.2 \end{bmatrix}$$

$$B = [0 \quad 0 \quad 1 \quad 0]^T$$

In the subsequent subsection, the OLKF, modified NE and modified LGA are implemented to the above MSD system and results are discussed in the following paras. In the modified NE and LGA schemes, the missing measurement samples are reconstructed by using linear prediction schemes and hence the estimation error is less as compared to OLKF when data loss occurs.

*B. Simulation and Results*

In the simulations, the results obtained for OLKF, modified NE and LGA with the data loss are compared. The sampling time period is $T_s = 1ms$. Data loss occurs from 2.4 to 2.7 s. In all results, the solid line shows the actual state and the dashed line shows the estimated state.

In Fig. 3, the four associated states are shown for the open-loop scheme along with actual states. It can be seen that during the data loss period, OLE betrays from the actual state track prominently. The data loss region is highlighted in Fig. 4.

This is because, no update step is performed in OLKF in the event of data loss, and therefore it is computationally less expensive. After performing simulation for 100 times on a system Core i3-3110M CPU @ 2.40GHz - 2.40GHz, RAM 2.00GB, 64-bit Operating System; the mean time taken by OLKF is 2.4128 s in case of data loss from 2.3 to 2.8 s.

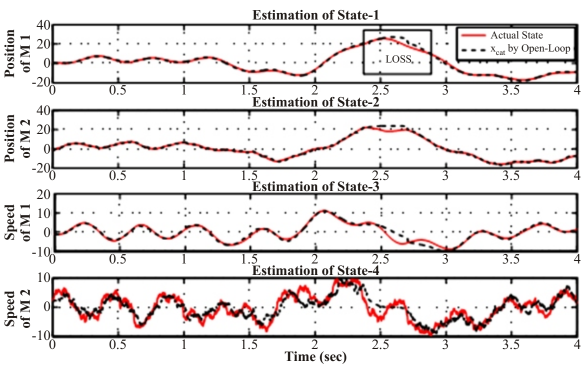


Fig. 3. Performance of Open Loop Estimation

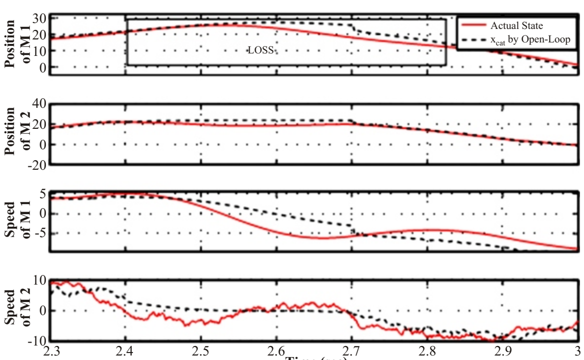


Fig. 4. Highlighted view of data loss region

The estimated four states associated with MSD system are obtained using modified NE method. In order to grasp clear taste of the modified LGA scheme, only 2[nd] state (for which measurement data is available) has been analyzed in this paper. Its estimation through modified NE method is shown in Fig 5. It can be verified that modified NE method, at the cost of computational efforts, provides better estimation results than OLE scheme.

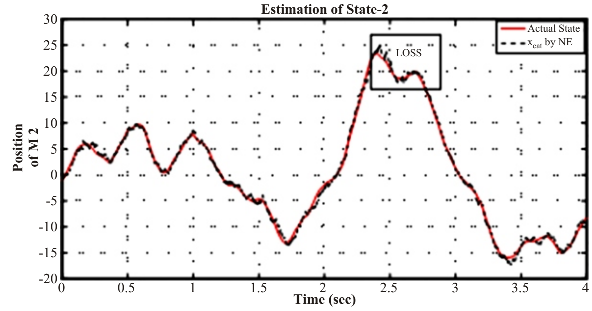


Fig. 5. Estimation of Position for Mass 2 through modified NE

Since in modified NE, the LPCs are computed using inversion of large matrices (of order $M \times M$, where $M$ is decided by Algorithm 3), therefore it is computationally expensive.

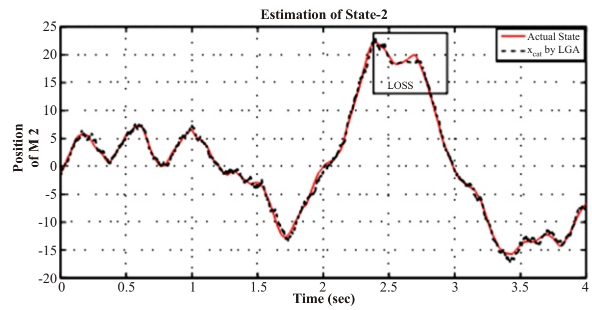


Fig. 6: Performance of LGA

Similarly, the estimation of state 2, shown in Fig. 6, has been achieved using modified LGA, which is the main contribution of this paper. It can be seen that the estimated result (dashed line) of the modified LGA tracks the actual position (solid line) and does not diverge significantly as compared to OLE and modified NE methods. Since LGA avoids the inversion of large matrices, it is computationally inexpensive than NE method. This claim can be verified from Table I. In addition, modified LGA also overcomes the issue of large dynamics range in a fixed point environment which is raised in LDA scheme.

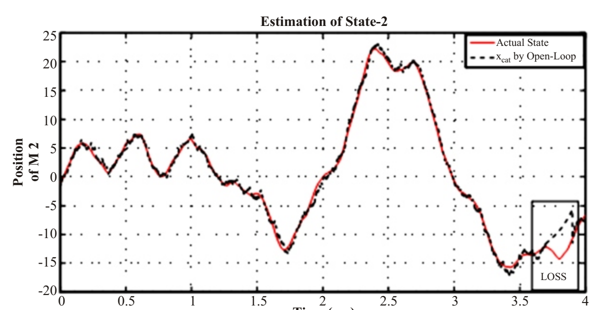


Fig. 7: Open Loop Estimation for data loss from 3.7 3.9 sec

In order to view broader spectrum of modified scheme, LGA is tested for data loss at various locations. In this connection, Figures 7-11 show the performance of OLE, modified NE and modified LGA scheme respectively.

It has been considered necessary to present error analysis for the three under discussion schemes (OLE, modified NE and modified LGA), Fig. 10 shows the absolute error analysis. Truly speaking, modified NE and LGA provide significantly small error than OLE during the data loss period.

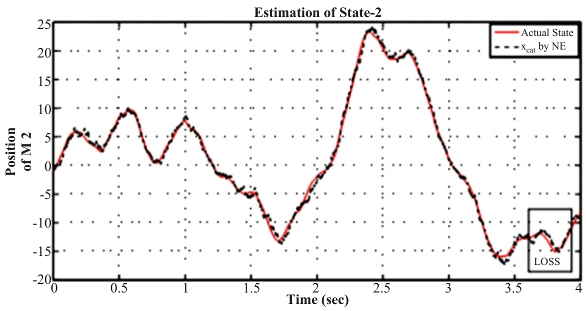


Fig. 8. Estimation through NE for data loss 3.7 - 3.9 sec

Since this paper focuses on the performance of modified LGA method compare to OLE and NE, Table I and Fig. 11 describe the cumulative computational time for modified NE and LGA schemes in tabulated form and graphically. As discussed in previous Figures, OLE abruptly diverges and hence suffers from large errors. Normal Equation method, on the other hand, is computationally expensive. The modified LGA overcomes both of these problems and hence provide better estimation results.

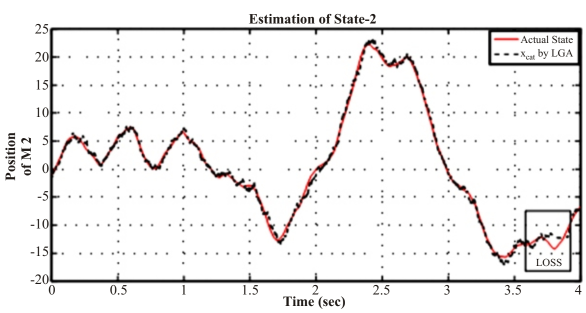


Fig. 9. Performance of LGA for data loss 3.7  3.9 sec

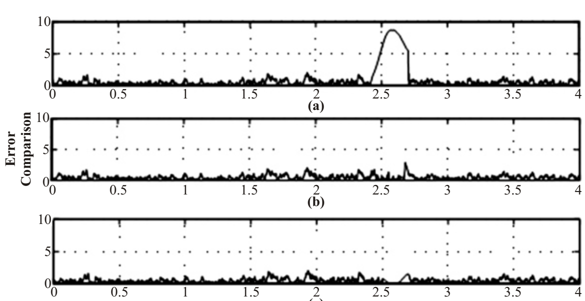


Fig. 10: Error Comparison a) Open-Loop Estimation, b) Modified NE, c) Modified LGA

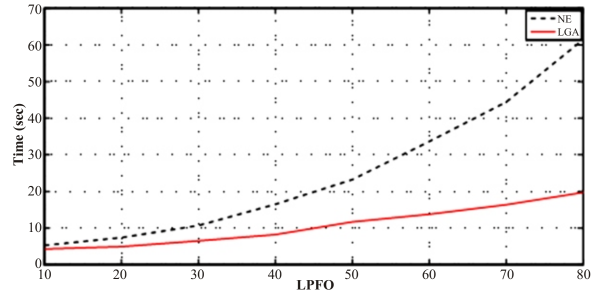


Fig. 11. Computational time comparison of NE and LGA for different LPFO

TABLE I
COMPUTATION ANALYSIS FOR NORMAL EQUATION AND MODIFIED LGA

| LPFO | Comp. Time for NE (s) | Comp. Time for LGA (s) |
|---|---|---|
| 10 | 5.2833 | 4.2774 |
| 20 | 7.3977 | 4.9342 |
| 30 | 10.6921 | 6.5057 |
| 40 | 16.4922 | 8.2134 |
| 50 | 23.2320 | 11.6752 |
| 60 | 33.6234 | 13.7644 |
| 70 | 44.3985 | 16.3645 |
| 80 | 61.6001 | 19.7254 |

## VI. CONCLUSION

Earlier methods adopted to deal data loss scenarios in state estimation have shortcomings such as more computational time and large errors. OLKF takes much less time but it has unbounded error. Alternatively NE method reduces the error but it takes more computational time in calculating LPCs. In this paper, a modified version of a particular Linear Prediction technique namely, LerouxGueguenAlgorithm has been presented to handle these two shortcomings. The proposed scheme avoids the inversion of large matrices, hence is computationally effective. LGA also improves the performance of OLKF by bounding the error during the data loss. A minimum mean square error based criteria is set to decide the order of linear prediction filter. Hence LGA is considered an optimal technique to compute LPCs in case of data loss.

## CONFLICT OF INTEREST

The authors declare that there is no conflict of interest regarding the publication of this article.

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